

Integral harmonic sums?

Theorem. For all $n > 1$, $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ isn't an integer.

Proof. Suppose there exist an integer $n > 1$ such that $H_n = k \in \mathbb{N}$.

We have $kn! = \frac{n!}{1} + \frac{n!}{2} + \cdots + \frac{n!}{n}$, which is a sum of integers.

Let p be the largest prime number of $\{1, \dots, n\}$. Let's show that $v_p(n!) = 1$.

Suppose $p^2 \mid n!$. Then there must be another multiple of p in $\{1, \dots, n\}$

Let's note it lp with $l \geq 2$ an integer.

We have $n \geq lp \geq 2p$. According to Bertrand's postulate, there exist a prime number q such that $p < q < 2p$.

Therefore, $q > p$ and $q < 2p \leq n$, which contradicts the fact that p is the largest prime number of $\{1, \dots, n\}$.

So $v_p(n!) = 1$ and, consequently, one and only one term among

$$\frac{n!}{1}, \frac{n!}{2}, \dots, \frac{n!}{n}$$

isn't divisible by p : $\frac{n!}{p}$.

But $p \mid n! \mid \frac{n!}{1} + \frac{n!}{2} + \cdots + \frac{n!}{n}$ so p must divide $\frac{n!}{p}$, which is impossible.

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